Part VII

Dynamic

Dynamic Analysis

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Dynamic Analysis

Dynamic loads can develop in the ground and underground structures due to earthquakes, machinery motions, pile driving, explosions, etc. The effects and characteristics of these loads on the ground or underground structures are as follows:

- Earthquake: An earthquake has the greatest influence on the ground and underground structures compared to any other dynamic loads. An earthquake causes settlements of structures' foundations, vertical mis-alignment of structures and liquefaction that leads to inability to support structures or even floatation of some light structures. An earthquake generates random motions in any direction, and it is almost impossible to predict when and where an earthquake may occur.
- Machinery motions: Dynamic loads due to machinery are generated by reciprocation or rotation of machines, which are characterized by harmonic loads defined as sine waves.
- Pile driving or explosions: These dynamic loads are normally generated during construction and are referred to as impact loads, which generate significant energy for a short period. Like seismic loads, impact loads are not periodic.

Figure A.1 Damage from earthquakes

Ground response analysis is performed in order to generate the design response spectrum, to establish the dynamic stress-strain relationship of the ground and to calculate the seismic loads acting on the ground structures due to an earthquake. The characteristic of the seismic wave at a bedrock outcrop is almost similar to that at bedrock. However, the characteristic of a seismic wave in soft soil or at the ground surface of deep ground can widely vary depending upon the soil properties. The dynamic behavior of a structure founded on soft soil can be greatly affected by soil-structure interaction. If a structure is founded on rock whose shear strain is less than 10^{-6} with the velocity of shear wave greater than 1,100m/sec, the soil-structure interaction can be ignored.

Soil-structure interaction can be summarized to have largely two effects. The first one is the change in dynamic characteristics of the structure. Because the stiffness of soil is normally smaller than that of the structure, the stiffness and the natural frequency of the structure become smaller. The second one pertains to increasing the damping of the structural system due to the addition of radiation damping. These effects vary depending on the composition of soil layers, material properties, specified seismic data and the frequency of the structure. Therefore, the response due to soil-structure interaction analysis can be larger or smaller than the general seismic analysis results with the assumption of fixity at the ground.

Soil-structure interaction due to an earthquake excitation can be broadly divided into kinematic interaction and inertial interaction. In an earthquake, the phenomenon that the behavior of ground in which a massless structure is situated is different from the behavior of the ground without the presence of the structure is known as kinematic interaction. Since an incident wave with a high order frequency component retains a short wavelength, the stiffness of the foundation absorbs and reflects the wave. But an incident wave with a low order frequency component retains a long wavelength, which passes through the structure and generates motion in the structure. In case of a vertically incident shear wave, rocking motion is generated in the structure. The ground motion generated by kinematic interaction is also referred to as scattering motion. Kinematic interaction is generally affected by the type of incident wave and the shape of foundation. If the structure is on the ground surface, the foundation is embedded in the ground at a shallow depth, or the surface wave is not considered, the effect of kinematic interaction is not so significant that it may be ignored in seismic analysis. When the structure is excited by a seismic wave, the inertia force is generated by the structure's mass. The inertia force in turn generates the kinematic energy of the structure, which is then radiated to the far field soil resulting in radiation motion of the ground. Inertial interaction increases as the stiffness difference between the

structure and the nearby ground increases. Therefore, when a very stiff structure is situated in soft soils, the influence of inertia interaction is much greater.

Soil-structure interaction analysis methods are largely classified into ‘substructure method’ and ‘direct method’. The substructure method undergoes a series of analyses by steps whose results are compiled in order to obtain the final response.

- Step1: From the controlled motion in the free field, determine the input motion at the soil-structure interface considering the soil properties and the effect of foundations.
- Step2: Define the dynamic stiffness of the semi-infinite ground at the soil-structure interface.
- Step3: Model a soil-structure system by combining the dynamic soil stiffness calculated from Step2 with the structure, and calculate the structural response by applying the excitation of the input motion obtained from Step1.

Even though it is difficult to fully consider the nonlinear properties of soil, a most appropriate analytical method can be chosen for each step. It is also possible to evaluate results at each step, which allows evaluating the sensitivity. Analysis by this method is more economical than by the direct method.

The direct method uses the finite element method or finite difference method to model the structure and the nearby ground of a certain limit together to analyze the soil-structure interaction at a time. The direct method takes much longer analysis time than the substructure method, but it has the advantage of being able to consider nonlinear compatibility condition at the soil-structure interface, the nonlinearity of soil, etc. The direct method in SoilWorks supports time history analysis and equivalent linear analysis.

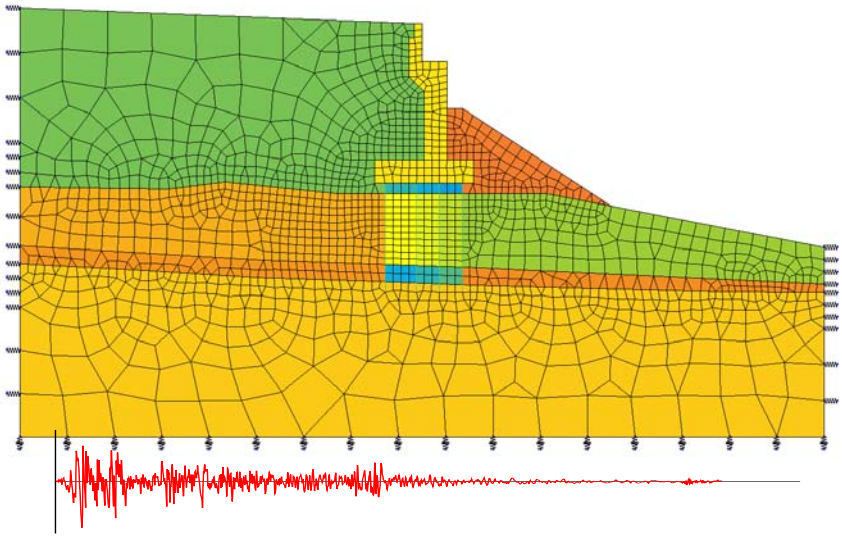


Figure A.2 Dynamic analysis model of abutment and ground using finite elements subjected to seismic loading

Chapter 1

Eigenvalue Analysis

Eigenvalue analysis is used to analyze the dynamic characteristics of soil or underground structures, which is in turn used to predict the characteristics or the trend of dynamic responses of the soil or underground structures. Using the orthogonality relationship of separating the modes, mode superposition and response spectrum can be used to easily predict the dynamic responses of soil or underground structures. Such reasons lead to a wide use of eigenvalue analysis in dynamic analysis.

SoilWorks provides 'Subspace iteration method' and 'Lanczos method' for eigenvalue analysis. Chapter 1.2 and 1.3 further describes the two methods in detail.

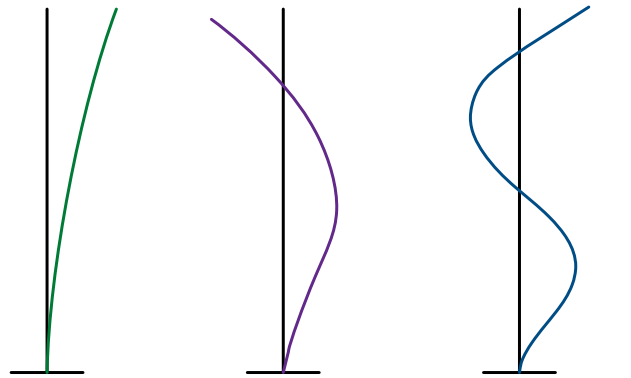
1.1 Eigenvector Analysis

In SoilWorks, mode shapes and natural periods of an undamped free vibration are obtained from the characteristic equation below.

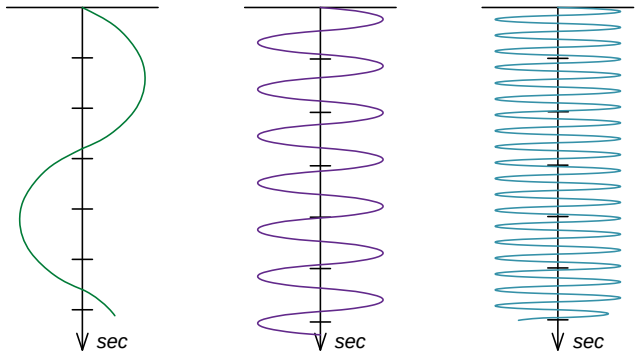
$$\mathbf{K}\underline{\Phi}_n = \lambda_n \mathbf{M}\underline{\Phi}_n \qquad (1.1.1)$$

where,

$\mathbf{K}$	: Stiffness matrix of ground or underground structure
$\mathbf{M}$	: Mass matrix of ground or underground structure
λ_n	: n-th mode eigenvalue
$\underline{\Phi}_n$	: n-th mode shape (mode vector)



(a) Mode shapes



$\lambda_1 = 1.87510407$ $\lambda_2 = 4.69409113$ $\lambda_3 = 7.85475744$

$T_1 = 1.78702$ sec $T_2 = 0.28515$ sec $T_3 = 0.10184$ sec

(b) Natural periods

Figure 1.1.1 Mode shapes and corresponding natural periods of a prismatic cantilever beam

Eigenvalue analysis is also referred to as “free vibration analysis”, which is used to analyze the dynamic characteristics of ground or underground structures.

The dynamic characteristics obtained by an eigenvalue analysis for ground or an underground structure include vibration modes (mode shapes), natural periods of vibration

(natural frequencies) and modal participation factors. They are determined by the mass and stiffness of ground or an underground structure.

Vibration modes take the form of natural shapes in which a structure or ground freely vibrates or deforms. The first mode shape (or fundamental vibration shape) refers to a deformed shape by the least energy or force. The shapes formed with increase in energy define the subsequent higher modes. Figure (1.1.1) shows the vibration modes of a cantilever beam, arranged in the order of increasing energy requirements for the respective deflected shapes, starting from the shape formed by the least energy.

A natural period of vibration refers to the time required to complete one full cycle of the free vibration motion in the corresponding natural mode.

The method of calculating the natural periods in a single degree of freedom system (SDOF) assumes that the loading and damping terms are equal to zero. The equation of motion in SDOF yields a free vibration equation of a linear second order differential equation as given in Eq. (1.1.2).

$$\begin{aligned} m\ddot{u} + c\dot{u} + ku &= p(t) \\ m\ddot{u} + ku &= 0 \end{aligned} \qquad (1.1.2)$$

Since u is the displacement due to vibration, and if $u = A\cos\omega t$ is assumed, where, A is a constant related to the initial displacement, then Eq. (1.1.2) can be written as,

$$(-m\omega^2 + k)A\cos\omega t = 0 \qquad (1.1.3)$$

In order to satisfy Eq. 1.1.3, the value in the parenthesis to the left must be zero, which leads to Eq. (1.1.4)

$$\omega^2 = \lambda = \frac{k}{m}, \quad \omega = \sqrt{\frac{k}{m}}, \quad f = \frac{\omega}{2\pi}, \quad T = \frac{1}{f} \qquad (1.1.4)$$

where,

ω^2 : Eigenvalue

ω : Rotational natural frequency

f : Natural frequency

T : Natural period

The modal participation factor is expressed as a ratio of contribution by the corresponding mode to the contribution by the total modes and is written as,

$$\tau_m = \frac{\sum M_i \varphi_{im}}{\sum M_i \varphi_{im}{}^2} \tag{1.1.5}$$

where,

τ_m : Modal participation factor,

m : Mode number,

M_i : Mass at location i,

φ_{im} : m-th mode shape at location i

Most seismic design codes stipulate that the sum of the effective modal masses included in an analysis should be greater than 90% of the total mass. This will ensure that most critical modes affecting the results are included in the design.

$$M_m = \frac{\left[\sum \varphi_{im} M_i\right]^2}{\sum \varphi_{im}{}^2 M_i} \tag{1.1.6}$$

where,

M_m : Effective modal mass

If certain degrees of freedom of a given mass are constrained, the mass will be included in the total mass but excluded from the effective modal mass due to the restraints on the corresponding mode vectors. Accordingly, if the user attempts to evaluate the ratio of the effective modal mass to the total mass, the degrees of freedom pertaining to the mass components must not be constrained.

In order to accurately analyze the dynamic behavior of ground or an underground structure, the analysis must closely reflect the mass and stiffness, which are most important factors for determining the eigenvalues. In most cases, finite element models can closely reflect

the stiffness components of structural members. In the case of mass however, the user is required to accurately assess the masses pertaining to the self-weights of structural components, which are relatively small compared to the total mass. Inclusion of masses for those materials not included in the analysis model must be accounted for in eigenvalue analysis.

Mass components are generally specified by 3 translational masses and 3 rotational mass moments of inertia consistent with 6 degrees of freedom per node. The rotational mass components pertaining to the rotational mass moments of inertia do not directly affect the dynamic response of a structure because an earthquake is imposed in the form of translational ground accelerations in seismic design. However, when the ground or the structure is of an irregular configuration, where the mass center does not coincide with the stiffness center, the rotational mass moments of inertia indirectly affect the dynamic response by partially changing the mode shapes.

Mass components are calculated by the following equations: (See Table 1.1.1)

Translational mass : $\int dm$ (1.1.7)

Rotational mass moment of inertia : $\int r^2 dm$

where,

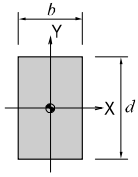
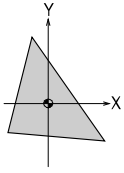
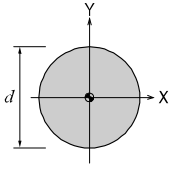
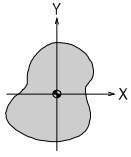
r : distance from the center of the total mass to the center of a corresponding infinitesimal mass.

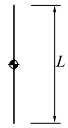
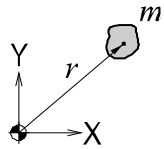
The unit for mass is defined by the weight divided by the acceleration of gravity, Weight (Time^2/Length). The unit for rotational mass moment of inertia is defined by the mass multiplied by the square of a length unit, Weight (Time^2/Length)$\times$Length^2. For example, if the MKS or English unit system is used, the mass is determined by the weight divided by the acceleration of gravity. In the case of an SI unit system the mass is determined by directly using the weight in the MKS system. Nevertheless, values used in the MKS system must be multiplied by the acceleration of gravity for defining the stiffness or loading in the SI unit system.

In SoilWorks, lumped mass is used for efficiency in the analysis process. Mass data can be specified using ‘Loads | Boundaries | Analysis > Static Load > Nodal Mass’ from the main menu.

Table 1.1.1 Calculations for Mass data

ρ : mass per unit area ⊕ : mass center

Shape	Translation mass	Rotational mass moment of inertia
Rectangular shape 	$M = \rho bd$	$I_m = \rho\left(\frac{bd^3}{12} + \frac{db^3}{12}\right)$ $= \frac{M}{12}\left(b^2 + d^2\right)$
Triangular shape 	$M = \rho \times$ area of triangle	$I_m = \rho\left(I_x + I_y\right)$
Circular shape 	$M = \rho\left(\frac{\pi d^2}{4}\right)$	$I_m = \rho\left(\frac{\pi d^4}{32}\right)$
General shape 	$M = \rho \times \int dA$	$I_m = \rho\left(I_x + I_y\right)$

Line shape 	ρ_L = Mass per unit length $M = \rho_L \times L$	$I_m = \rho_L \left(\frac{L^3}{12} \right)$
Eccentric mass 	Eccentric mass: m M = m	Rotational mass moment of inertia about its mass center: I_o $I_m = I_o + mr^2$

1.2 Subspace Iteration Method

The subspace iteration method pertains to iterative calculations of a number (N_s) of vectors (X_k) constituting the subspace, E_k to converge to natural modes $\left[\phi_1 \quad \phi_2 \quad .. \quad \phi_{N_s}\right]$. The following shows the process of calculating the vectors, X_k through a k number of iterations.

- For the first iteration, N_s number of initial vectors, X_1 constituting the subspace, E_1
- For the k-th iteration,
- Solution to linear simultaneous equations $KY_k = MX_k$
- Projection of stiffness matrix $K_{k+1} = Y_k^T KY_k$
- Projection of mass matrix $M_{k+1} = Y_k^T MY_k$
- Solution to projected eigenvalues $K_{k+1}Q_{k+1} = M_{k+1}Q_{k+1}\Lambda_{k+1}$

$$\mathbf{Q}_{k+1} = \left[\psi_1^* \quad \psi_2^* \quad \ldots \quad \psi_{N_s}^*\right]$$

$$\mathbf{\Lambda}_{k+1} = \begin{bmatrix} \lambda_1^* & & & \\ & \lambda_2^* & & \\ & & \ddots & \\ & & & \lambda_{N_s}^* \end{bmatrix}$$

- Calculation of X_{k+1} $\mathbf{X}_{k+1} = \mathbf{Y}_k \mathbf{Q}_{k+1}$

The λ_n^* and $\mathbf{X}_k$ that are computed from the above iterative process will converge to eigenvalues and natural modes respectively.

$$\lambda_m^* \to \lambda_m, \quad \mathbf{X}_k \to \left[\phi_1 \quad \phi_2 \quad \ldots \quad \phi_{N_s}\right] \tag{1.2.1}$$

When X_k consists of N_s number of vectors, the dimensions of K_k and M_k are fixed to $N_s \times N_s$ irrespective of the number of iterations. The convergence of the eigenvalues $\lambda_m^{(k+1)}$ computed from k iterations will be evaluated by the change in consecutive values as follows:

$$\left\| \frac{\lambda_m^{(k+1)} - \lambda_m^{(k)}}{\lambda_m^{(k+1)}} \right\| \le \varepsilon \tag{1.2.2}$$

The subspace method in SoilWorks allows the user to input the dimension of subspace $\mathbf{X}_k$, N_s, the maximum number of iterations, N_I, and the convergence tolerance, ε of Eq. (6.1.2). The dimension of subspace, N_s actually used in calculation is shown below. N_{s_0} is the subspace dimension specified by the user, N_s, and N_f is the number of eigenvalues to be calculated.

$$N_s = \max\{N_{s_0}, \min(2N_f, N_f + 8)\} \tag{1.2.3}$$

The maximum number of iterations in subspace iterative calculation is set to $N_I = 20$, and the convergence tolerance is set to $\varepsilon = 1.0 \times 10^{-10}$.

1.3 Lanczos Iteration Method

The Lanczos method is an iterative method of obtaining approximate eigenvalues using the tridiagonal matrix, $\mathbf{T}_k$ generated through the process of creating the Krylov's subspace $span(\mathbf{V}_{1,}\mathbf{V}_2,...,\mathbf{V}_k)$. In order to efficiently apply the Lanczos method to the eigenvalue problem of Eq. (6.1.1) stemming from the vibration analysis of ground or a underground structure, λ_m must be substituted with $\lambda_m = \sigma + 1/\theta_m$. This is referred to as the Shift-invert technique, and σ is the expected first eigenvalue. The Lanczos iteration method using the shift-invert technique is summarized as follows:

- For the first iteration, assume the initial block vector V_1
- For the k-th iteration
- Multiplication of mass matrix $\mathbf{U}_k = \mathbf{M}\mathbf{V}_k$
- Solution to linear simultaneous equations $(\mathbf{K} - \sigma\mathbf{M})\mathbf{W}_k = \mathbf{U}_k$
- Orthogonalization $\mathbf{W}_k^* = \mathbf{W}_k - \mathbf{V}_{k-1}\mathbf{B}_{k-1}^T$
- Calculation of matrix $\mathbf{C}_k$ $\mathbf{C}_k = \mathbf{V}_k\mathbf{M}\mathbf{W}_k^*$
- Orthogonalization $\mathbf{W}_k^{**} = \mathbf{W}_k^* - \mathbf{V}_k\mathbf{C}_k$
- Normalization within block vector $\mathbf{W}_k^{**} = \mathbf{V}_{k+1}\mathbf{B}_k$

SoilWorks uses the block vector, $\mathbf{V}_m$ to efficiently calculate eigenvalues, which is called the block Lanczos method. The block tridiagonal matrix, T_k generated from the process of the above iterative calculation is given as,

$$\mathbf{T}_k = \begin{bmatrix} \mathbf{C}_1 & \mathbf{B}_1^T & & & \\ \mathbf{B}_1 & \mathbf{C}_2 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \mathbf{C}_{k-1} & \mathbf{B}_{k-1}^T \\ & & & \mathbf{B}_{k-1}^T & \mathbf{C}_k \end{bmatrix} \tag{1.3.1}$$

Once the eigenvalue problem, $\mathbf{T}_k\psi_m^* = \theta_m^*\psi_m^*$ is solved using $\mathbf{T}_k$, λ_m^* can be obtained using $\lambda_m^* = \sigma + 1/\theta_m^*$, and λ_m^* is an approximate solution to the original eigenvalue problem

of Eq. (1.1.1). When the block dimension of $\mathbf{V}_k$ is set to N_b, the dimension of $\mathbf{T}_k$ increases to N_b, and λ_m^* converges to λ_m as the number of iterations increases. The approximate value, ϕ_m^* of the natural mode, ϕ_m is obtained from the following equation, which converges together with λ_m^*.

$$\phi_m^* = \begin{bmatrix} \mathbf{V}_1 & \mathbf{V}_2 & \mathbf{V}_3 & \dots & \mathbf{V}_k \end{bmatrix} \psi_m^* \tag{1.3.2}$$

The convergence of eigenvalues and natural modes calculated by the Lanczos method can be examined by the following:

$$\frac{\left\| \mathbf{K}\phi_m^* - \lambda_m^* \mathbf{M}\phi_m^* \right\|}{\left\| \mathbf{K} \right\|} \le \varepsilon \tag{1.3.3}$$

where, $\|\cdot\|$ is 2-norm, and SoilWorks uses $\varepsilon = 2.22 \times 10^{-16}$.

In the Lanczos method in SoilWorks, the range of interest in frequency to be calculated can be determined by two frequencies, f_1 and f_2. The range of interest in frequency can be reflected in the calculation by setting the expected eigenvalue to $\sigma = (2\pi f_1)^2$ by the shift-invert technique. Figure 1.3.1 shows the range and order of eigenvalues calculated by the relationship between f_1 and f_2. If the total number of eigenvalues to be calculated is set to N_f, the eigenvalues can be calculated depending on the magnitudes of f_1 and f_2 as follows:

- For $f_1 = f_2$: calculate N_f frequencies close to f_1
- For $f_1 < f_2$: calculate N_f frequencies close to f_1 in the range of $[f_1, f_2]$
- For $f_1 > f_2$: calculate N_f frequencies close to f_1 in the range of $[f_2, f_1]$

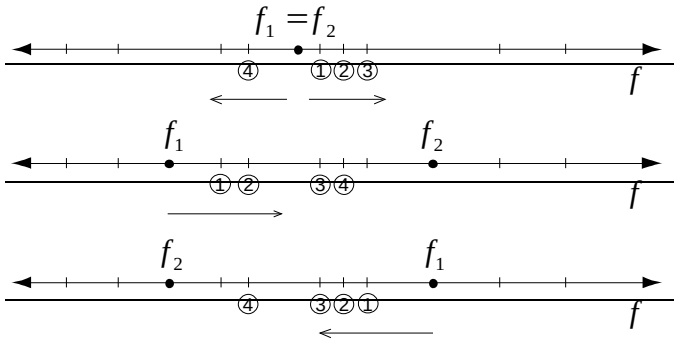


Figure 1.3.1 Direction of searching frequencies

The default values for the range of interest in frequency, f_1 and f_2 are $f_1 = 0,\ f_2 = 1600$.

Chapter 2

Damping

2.1 Introduction

Damping in ground or an underground structure in dynamic analysis can be largely classified into the following:

A. Viscous damping (Voigt model and Maxwell model)

B. Hysteretic damping

C. Friction damping

1) Internal friction damping (Material damping)
2) External friction damping
3) Sliding friction damping

D. Radiation damping

E. Modal damping

1) Proportional damping
   - ✓ Mass proportional type
   - ✓ Stiffness proportional type
   - ✓ Rayleigh type
   - ✓ Caughey type

2) Non-proportional damping
   - ✓ Energy proportional type

Among the many different ways of expressing damping above, modal damping is most frequently used in the numerical analysis of structures.

Modal damping is determined for each modal natural frequency of a vibration system. The modal damping can be classified into proportional and non-proportional damping. SoilWorks provides proportional damping, which includes mass proportional, stiffness proportional and Rayleigh type damping.

2.2 Proportional Damping

The mass-proportional damping represents external viscous damping such as caused by air resistance, which is based on the assumption that damping matrix is proportional to mass. The stiffness-proportional damping on the other hand may overestimate the damping of high order modes since it is assumed that damping is proportional to stiffness due to the difficulty in directly considering the radiation damping (vibration energy release into the ground of infinite nature).

Rayleigh damping is a type of stiffness-proportional damping with the modification of the damping coefficients for high order modes, which can be expressed by combining the mass and stiffness types of proportional damping.

The general type of proportional damping matrix C is defined by Caughey as follows:

$$C = M\{\sum_{j=0}^{N-1} a_j (M^{-1}K)^j\} \tag{2.2.1}$$

where,

j, N : Degrees of freedom of nodes (Mode number)

In Eq. (2.2.1), $M^{-1}K$ can be obtained from the equation of free vibration of an undamped system as below. Asssuiming,

$$M\{y\} + K\{y\} = 0 \tag{2.2.2}$$

$$\{y\} = \{u\}e^{iax} \tag{2.2.3}$$

Substituting into Eq. (2.2.1), the following is obtained:

$$(-\omega^2 M + K)\{u\} = \{0\} \tag{2.2.4}$$

From Eq. (2.2.4), $M^{-1}K = \omega^2$ is obtained. Here, ω^2 exists in as many numbers as the number of modes. Considering all the modes, it is expressed as ${}_s\omega^2$.

Substituting $M^{-1}K$ obtained from Eq. (2.2.2) - (2.2.3) into Eq. (2.2.1), and multiplying $\{ {}_s u \}^T$ on the left hand side of and $\{ {}_s u \}$ on the right hand side, Eq. (2.2.5) derived.

$$\{ {}_s u \}^T C \{ {}_s u \} = {}_s C = \sum_{j=0}^{N-1} a_j \cdot {}_s\omega^{2j} \cdot \{ {}_s u \}^T M \{ {}_s u \} = \sum_{j=0}^{N-1} a_j \cdot {}_s\omega^{2j} \cdot {}_s M \tag{2.2.5}$$

Also, the damping coefficient for Mode s, ${}_s h$ can be expressed as,

$${}_s C = 2\,{}_s h \cdot {}_s\omega \cdot {}_s M \tag{2.2.6}$$

The damping coefficient, ${}_s h$ in Eq. (2.2.5) and (2.2.6) is,

$$\begin{aligned} {}_s h &= \frac{{}_s C}{2\,{}_s\omega \cdot {}_s M} = \frac{1}{2\,{}_s\omega} \sum a_j \cdot {}_s\omega^{2j} \\ &= \frac{1}{2}\left(\frac{a_0}{{}_s\omega} + a_1 \cdot {}_s\omega + a_2 \cdot {}_s\omega^3 + \cdots + a_{N-1} \cdot \omega^{2N-3} \right), \quad s = 1 - N \end{aligned} \tag{2.2.7}$$

The damping coefficients for N number of natural modes are then individually determined.

The damping coefficients and matrices for the mass proportional type and stiffness proportional type are as follows:

$${}_s h = \frac{a_0}{2\,{}_s\omega}, \quad C = a_0 M \quad \text{: Mass proportional type} \tag{2.2.8}$$

$${}_s h = \frac{a_1 \cdot {}_s\omega}{2}, \quad C = a_1 M \quad \text{: Stiffness proportional type} \tag{2.2.9}$$

In the case of the Rayleigh type, the above becomes,

$${}_s h = \frac{1}{2}\left(\frac{a_0}{{}_s\omega} + a_1 \cdot {}_s\omega \right), \quad C = a_0 M + a_1 K \tag{2.2.10}$$

where,

$$a_0 = \frac{2 \cdot {}_1\omega \cdot {}_2\omega ({}_1h \cdot {}_2\omega - {}_2h \cdot {}_1\omega)}{({}_2\omega^2 - {}_1\omega^2)}$$

$$a_1 = \frac{2({}_2h \cdot {}_2\omega - {}_1h \cdot {}_1\omega)}{({}_2\omega^2 - {}_1\omega^2)}$$

Chapter 3

Response Spectrum Analysis

The dynamic equilibrium equation for ground or an underground structure subjected to seismic loading used in response spectrum analysis can be expressed as follows:

$$[M]\ddot{u}(t)+[C]\dot{u}(t)+[K]u(t)=-[M]w_g(t) \tag{3.1}$$

where,

$[M]$: Mass matrix,

$[C]$: Damping matrix,

$[K]$: Stiffness matrix,

w_g : Ground acceleration

and, *u(t)*, $\dot{u}(t)$, $\ddot{u}(t)$ are relative displacement, velocity and acceleration respectively.

Response spectrum analysis assumes the response of a multi-degree-of-freedom (MDOF) system as a compound of single-degree-of-freedom (SDOF) systems. A response spectrum defines the peak values of responses (acceleration, velocity & displacement) corresponding to and varying with natural periods (or frequencies) of vibration, which have been obtained through a numerical integration process. Response spectrum analysis is generally carried out for seismic design using the design spectra defined in design standards.

In order to predict the peak design response values, the maximum response for each mode is obtained first, and then the maximum responses of the modes are combined by an appropriate method. For seismic analysis, the displacement and inertial force corresponding to a particular degree of freedom for the m-th mode are expressed as follows:

$$d_{xm}=\Gamma_m\varphi_{xm}S_{dm}\ ,\quad F_{xm}=\Gamma_m\varphi_{xm}S_{am}W_x \tag{3.2}$$

where,

Γm : m-th Modal participation factor

φ_m : m-th Modal vector at location x
S_{dm} : Normalized spectral displacement for m-th mode period
S_{am} : Normalized spectral acceleration for m-th mode period
W_x : Mass at location x

Linear interpolation is used in SoilWorks to find the spectral value corresponding to the calculated natural period for a given mode from the spectral data defined for a range of certain periods. It is therefore recommended that spectral data be provided at close increments of natural periods at the locations of significant changes in the spectrum curvature (refer to Figure 3.1). The range of natural periods for spectral data must be sufficiently extended to include the maximum and minimum natural periods obtained from the eigenvalue analysis.

SoilWorks can easily generate the spectral data for seismic design by specifying Dynamic coefficient, Foundation factor, Zoning factor, Importance factor, Ductility factor (or Response modification factor or Seismic response factor), etc.

SoilWorks permits response spectrum analysis in any direction on the Global X-Z plane and in the Global Z direction. The user may choose an appropriate method of modal combination for analysis results such as the Complete Quadratic Combination (CQC) method or the Square Root of the Sum of the Squares (SRSS) method.[1]

The following describes the methods for modal combinations:

- ✓ **SRSS (Square Root of the Sum of the Squares)**

$$R_{\max} = \left[R_1^2 + R_2^2 + \cdots + R_n^2 \right]^{1/2} \tag{3.3}$$

- ✓ **ABS (ABsolute Sum)**

$$R_{\max} = |R_1| + |R_2| + \cdots + |R_n| \tag{3.4}$$

[1] The user may reinstate the signs lost during the modal combination process and apply them to the response spectrum analysis results. For details, refer to “Analysis and Design > Analysis Control > Dynamic Module > Respnse Spectrum Analysisl” in the Online Manual.

✓ **CQC (Complete Quadratic Combination)**

$$R_{\max} = \left[\sum_{i=1}^{N} \sum_{j=1}^{N} R_i \rho_{ij} R_j \right]^{1/2} \tag{3.5}$$

where,

$$\rho_{ij} = \frac{8\xi^2 (1+r) r^{3/2}}{(1-r)^2 + 4\xi^2 r (1+r)^2},$$

$$r = \frac{\omega_j}{\omega_i},$$

$R_{\max}$: Peak response,

R_i : Peak response of i-th mode,

r : Ratio of natural frequency of j-th mode to i-th mode,

ξ : Damping ratio

When i = j in Eq. (3.5), then $\rho_{ij} = 1$ regardless of the damping ratio. If the damping ratio (ξ) becomes zero (0), both CQC and SRSS methods produce the same results.

The ABS method produces the largest combination values among the three methods. The SRSS method has been widely used in the past, but it tends to overestimate or underestimate the combination results in the cases where the values of natural frequencies are close to one another**.** As a result, the use of the CQC method is increasing recently as it accounts for probabilistic inter-relations between the modes.

For example, if the frequencies and displacements of individual modes are calculated as below for ground or an underground structure having 3 DOF with a damping ratio of 0.05, the results from the applications of SRSS and CQC are compared as follows:

✓ **Natural frequencies**

$$\omega_1 = 0.46, \quad \omega_2 = 0.52, \quad \omega_3 = 1.42$$

✓ **Maximum modal displacements: D_{ij} (displacement components of i-th degree of freedom for j-th mode)**

$$D_{ij} = \begin{vmatrix} 0.036 & 0.012 & 0.019 \\ -0.012 & 0.014 & -0.005 \\ 0.049 & 0.002 & -0.017 \end{vmatrix}$$

✓ **If SRSS is used to compute the modal combination for each degree of freedom,**

$$R_{\max} = \left[R_1^{\,2} + R_2^{\,2} + R_3^{\,2} \right]^{1/2} = \{0.042 \quad 0.046 \quad 0.052\}$$

✓ **If CQC is used,**

$$\rho_{12} = \rho_{21} = 0.3985$$

$$\rho_{13} = \rho_{31} = 0.0061$$

$$\rho_{23} = \rho_{32} = 0.0080$$

$$\begin{aligned} R_{\max} &= \left[R_1^{\,2} + R_2^{\,2} + R_3^{\,2} + 2\rho_{12}R_1R_2 + 2\rho_{13}R_1R_3 + 2\rho_{23}R_2R_3 \right]^{1/2} \\ &= \{0.046 \quad 0.041 \quad 0.053\} \end{aligned}$$

Comparing the two sets of displacements for each degree of freedom, we note that the SRSS method underestimates the magnitude for the first degree of freedom, but overestimates the value for the second degree of freedom relative to those obtained by CQC. Thus, the SRSS method should be used with care when natural frequencies are close to one another.

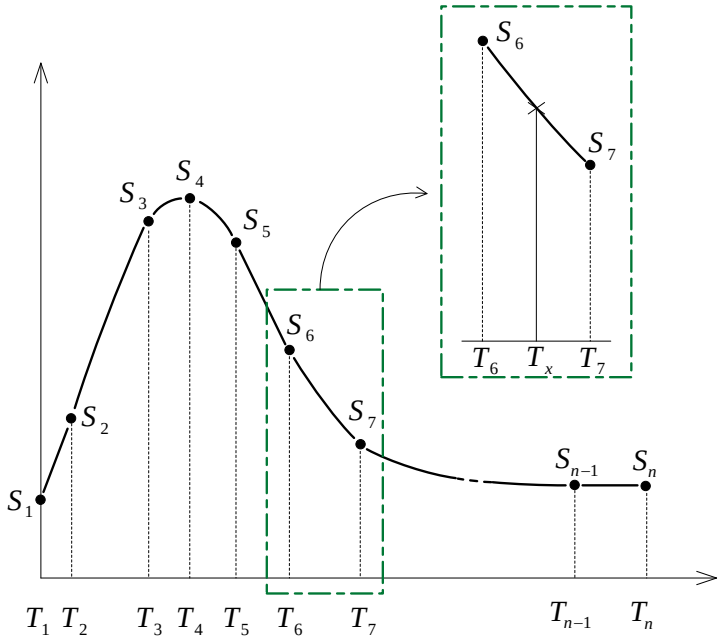


Figure 3.1 Response spectrum curve and linear interpolation of spectral data

Chapter 4

Time History Analysis

4.1 Introduction

The dynamic equilibrium equation for time history analysis is written as:

$$[M]\ddot{u}(t)+[C]\dot{u}(t)+[K]u(t)=p(t) \tag{4.1.1}$$

where,

$[M]$: Mass matrix

$[C]$: Damping matrix

$[K]$: Stiffness matrix

$p(t)$: Dynamic load

and, $u(t)$, $\dot{u}(t)$ and $\ddot{u}(t)$ are displacement, velocity and acceleration respectively.

Time history analysis seeks out a solution to the dynamic equilibrium equation when ground or an underground structure is subjected to dynamic loading. It calculates a series of ground and structural responses (displacements, member forces, etc.) within a given period of time based on the dynamic characteristics of the ground and structure under the applied loading. SoilWorks uses the modal superposition method and the direct integration method for time history analysis.

The following outlines the general concept of the modal superposition method and precautions when entering the data:

4.2 Modal Superposition Method

Modal superposition finds the displacements of ground or an underground structure in the form of a linear combination of displacement shapes having the orthogonality to one another. This method is used on the assumption that the damping matrix can be composed of a linear combination of the mass and stiffness matrices as presented below.

$$[C] = \alpha[M] + \beta[K] \tag{4.2.1}$$

$$\Phi^T M \Phi \ddot{q}(t) + \Phi^T C \Phi \dot{q}(t) + \Phi^T K \Phi q(t) = \Phi^T F(t) \tag{4.2.2}$$

$$m_i \ddot{q}_i(t) + c_i \dot{q}_i(t) + k_i q_i(t) = P_i(t) \quad (i = 1, 2, 3, \cdots, m) \tag{4.2.3}$$

$$u(t) = \sum_{i=j}^{m} \Phi_i q_i(t) \tag{4.2.4}$$

$$\begin{aligned} q_i(t) = {} & e^{-\xi_i \omega_i t}\left[q_i(0)\cos\omega_{Di} t + \frac{\xi_i \omega_i q_i(0) + q_i(0)}{\omega_{Di}} \sin\omega_{Di} t \right] \\ & + \frac{1}{m_i \omega_{Di}} \int_0^t P_i(\tau) e^{-\xi_i \omega_i (t-\tau)} \sin\omega_{Di}(t-\tau) d\tau \end{aligned} \tag{4.2.5}$$

where,

$\omega_{Di} = \omega_i \sqrt{1 - \xi_i^2}$,

$\alpha,\ \beta$	:	Rayleigh coefficients,
ξ_i	:	Damping ratio for i-th mode,
ω_i	:	Natural frequency of i-th mode,
Φ_i	:	i-th Mode shape,
$q_i(t)$	:	Solution to SDF equation by i-th mode

In time history analysis, the displacements of ground or an underground structure are determined by summing up the product of each mode shape and the solution to the corresponding modal equation of the SDF system as expressed in Eq. (4.2.5). Its accuracy depends on the number of modes used.

The modal superposition method is very effective, which is most widely used in linear dynamic analyses for large ground or underground structures. However, this method cannot be applied to nonlinear dynamic analysis or to the cases in which the damping matrix cannot be assumed as a linear combination of the mass and stiffness matrices due to the presence of damping devices.

The following outlines some precautions required during data entry when using the modal superposition method:

- ✓ **Total analysis time (or number of iterations)**

- ✓ **Analysis time interval**: The time interval used in analysis can greatly affect the accuracy of analysis results. The magnitude of the time interval must be selected to closely reflect the periods of higher modes of the ground or underground structure and the period of the applied loading. The analysis time interval directly influences the integral term in Eq. (4.2.4), and as such specifying an improper time interval may lead to inaccurate results. In general, one-tenth of the highest modal period under consideration is a reasonable value for the time interval. In addition, the time interval should be smaller than that of the applied loading.

$$\Delta t = \frac{T_p}{10} \tag{4.2.6}$$

where,

T_p : The highest modal period being considered

- ✓ **Modal damping ratios (or Rayleigh coefficients):** Values for determining the energy dissipation (damping) properties of ground or an underground structure (damping ratio of the total ground or the underground structure or for each mode)

- ✓ **Dynamic loads:** Dynamic loads are directly applied to the nodes of ground or an underground structure or foundations, which are expressed as a function of time. The forcing function must sufficiently represent the change in total loadings. Loading at an undefined time is linearly interpolated.

4.3 Direct Integration Method

The direct integration method solves the dynamic equilibrium equation by gradually integrating it over time steps. Integration takes place at every time step without changing the form of the equilibrium equation. Various methods can be used to obtain the solution.

SoilWorks uses the Newmark method for direct integration, which shows good convergence. The following are the basic assumptions and integration process:

$$ {}^{t+\Delta t}\dot{u} = {}^{t}\dot{u} + [(1-\delta)\,{}^{t}\ddot{u} + \delta\,{}^{t+\Delta t}\ddot{u}]\Delta t \tag{4.3.1} $$

$$ {}^{t+\Delta t}u = {}^{t}u + {}^{t}\dot{u}\Delta t + [(\frac{1}{2}-\alpha)\,{}^{t}\ddot{u} + \alpha\,{}^{t+\Delta t}\ddot{u}]\Delta t^{2} \tag{4.3.2} $$

${}^{t+\Delta t}\ddot{u}$ is determined from Eq. (4.3.2), which is then substituted into Eq. (4.3.1) to calculate ${}^{t+\Delta t}\dot{u}$. Thus, the expressions like Eq. (4.3.3) can be written in terms of displacement, velocity and acceleration at the previous step and displacement at the current.

$$ \begin{aligned} {}^{t+\Delta t}\ddot{u} &= f({}^{t+\Delta t}u\ ,\ {}^{t}u\ ,\ {}^{t}\dot{u}\ ,\ {}^{t}\ddot{u}) \\ {}^{t+\Delta t}\dot{u} &= f({}^{t+\Delta t}u\ ,\ {}^{t}u\ ,\ {}^{t}\dot{u}\ ,\ {}^{t}\ddot{u}) \end{aligned} \tag{4.3.3} $$

By substituting the values of Eq. (4.3.4) into the dynamic equilibrium equation like Eq. (4.3.5), the equation can be defined in terms of displacement, velocity and acceleration at the previous step and displacement at the current step. The displacement at the current step can be obtained from Eq. (4.3.6). Using the displacement at the current step and the values from the previous step, acceleration and velocity at the current step can be found as shown in Eq. (4.3.7). Damping is calculated using the ratios of the stiffness and mass as shown in Eq. (4.3.8).

$$ [M]\,{}^{t+\Delta t}\ddot{u} + [C]\,{}^{t+\Delta t}\dot{u} + [K]\,{}^{t+\Delta t}u = {}^{t+\Delta t}p \tag{4.3.4} $$

$$\left[[K]+a_0[M]+a_1[C]\right]{}^{t+\Delta t}u = {}^{t+\Delta t}p+[M](a_0{}^{t}u+a_2{}^{t}\dot{u}+a_3{}^{t}\ddot{u}) + [C](a_1{}^{t}u+a_4{}^{t}\dot{u}+a_5{}^{t}\ddot{u}) \tag{4.3.5}$$

$$[\hat{K}]{}^{t+\Delta t}u = {}^{t+\Delta t}\hat{p} \tag{4.3.6}$$

$$[\hat{K}] = [K]+a_0[M]+a_1[C]$$

$${}^{t+\Delta t}\hat{p} = {}^{t+\Delta t}p+[M](a_0{}^{t}u+a_2{}^{t}\dot{u}+a_3{}^{t}\ddot{u})+[C](a_1{}^{t}u+a_4{}^{t}\dot{u}+a_5{}^{t}\ddot{u})$$

$${}^{t+\Delta t}\ddot{u} = a_0({}^{t+\Delta t}u-{}^{t}u)-a_2{}^{t}\dot{u}-a_3{}^{t}\ddot{u}\,, \qquad {}^{t+\Delta t}\dot{u} = {}^{t}\dot{u}+a_6{}^{t}\ddot{u}+a_7{}^{t+\Delta t}\ddot{u} \tag{4.3.7}$$

where,

$$a_0 = \frac{1}{\alpha\Delta t^2}, \quad a_1 = \frac{\delta}{\alpha\Delta t}, \quad a_2 = \frac{1}{\alpha\Delta t}, \quad a_3 = \frac{1}{2\alpha}-1$$

$$a_4 = \frac{\delta}{\alpha}-1, \quad a_5 = \frac{\Delta t}{2}\left(\frac{\delta}{\alpha}-2\right), \quad a_6 = \Delta t(1-\delta), \quad a_7 = \delta\Delta t$$

a, δ : Newmark integration variables (Stable if $\alpha = 0.5, \delta = 0.25$)

Δt : Integration time step

$$[C] = a[K]+b[M] \tag{4.3.8}$$

where,

a, b : Proportional constants for stiffness and mass for calculation of damping

The direct integration method should be used in most cases involving nonlinearity of stiffness and damping. Because direct integration is performed at all the time steps, the analysis time becomes proportional to the total number of time steps. The following outlines some precautions required during data entry when using the direct integration method:

- ✓ **Total analysis time (or number of iterations)**

- ✓ **Time step:** The time interval used in analysis can greatly affect the accuracy of analysis results. The magnitude of the time interval must be selected to closely reflect the periods of higher modes of the ground or underground structure and the period of the applied loading. The analysis time interval directly influences the integral term in Eq. (4.3.7), and as such specifying an improper time interval may lead to inaccurate results. In general, one-tenth of the highest modal period under consideration is a reasonable value for the time interval. In addition, the time interval should be smaller than that of the applied loading.

$$\Delta t = \frac{T_p}{10} \tag{4.3.9}$$

where,

T_p : The highest modal period being considered

- ✓ The time interval must not be unnecessarily set to a small number since the analysis time increases in proportion to the number of time steps.

- ✓ **Definition of damping using stiffness and mass:** Damping is defined in proportion to stiffness and mass.

- ✓ **Time integration method:** Integration parameters necessary for using the Newmark method are entered. While a constant acceleration calculates a stable and converged solution in every condition, a linear acceleration may lead to divergence depending on the condition. Whenever possible, it is more appropriate to use the integration parameters corresponding to a constant acceleration.

- ✓ **Dynamic loads:** Dynamic loads are directly applied to the nodes of ground or an underground structure or foundations, which are expressed as a function of time. The forcing function must sufficiently represent the change in total loadings. Loading at an undefined time is linearly interpolated.

The following describes the fundamentals of dynamic analysis of ground or an underground structure:

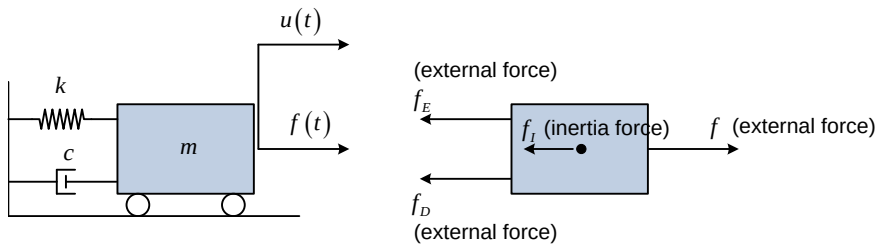
Figure (4.3.1) idealizes the motion of a SDOF structural system of ground or an underground structure. The equilibrium equation of motion subjected to forces acting on a SDOF system is as follows:

$$f_I(t) + f_D(t) + f_E(t) = f(t) \tag{4.3.10}$$

$f_I(t)$ is an inertia force, which represents the resistance to the change in velocity of the ground or underground structure. The inertia force acts in the direction opposite to the acceleration, and its magnitude is $m\ddot{u}(t)$.

$f_E(t)$ is an elastic force by which the ground or underground structure tends to restore its configuration to the original position after deformation has taken place. This force acts in the opposite direction to the displacement, and its magnitude is $ku(t)$.

$f_D(t)$ is a damping force, which is a fictitious internal force dissipating kinetic energy and thereby decreasing the amplitude of a motion. The damping force may develop in the form of internal friction. It acts in the opposite direction to the velocity, and its magnitude is $c\dot{u}(t)$.



(a) Idealized model (b) State of equilibrium

Figure 4.3.1 Motion of a SDOF System

The above forces are now summarized as,

$$f_I = m\ddot{u}(t)$$
$$f_D = c\dot{u}(t) \qquad (4.3.11)$$
$$f_E = ku(t)$$

where,

m	: Mass
c	: Damping coefficient
k	: Elastic coefficient

From the force equilibrium shown in Figure 4.3.1 (b), the equation of motion for a SDOF structural system is expressed as,

$$m\ddot{u} + c\dot{u} + ku = f(t) \qquad (4.3.12)$$

Eq. 4.3.12 becomes the equation of damped free vibration if f(t)=0, and it further becomes the equation of undamped free vibration if the condition of c=0 is additionally imposed. If f(t) is defined as an excitation force (or displacement, velocity, acceleration, etc.) relative to time, the equation then becomes to represent a forced vibration analysis problem. The solution can be determined by using the modal superposition method or the direct integration method.

Chapter 5

Equivalent Linear Analysis

5.1 1-Dimensional Dynamic Analysis

Free field analysis is referred to as the analysis for finding the response of the initial ground due to an earthquake input prior to constructing the structure in question. Free field analysis is mainly used to predict the vibration of ground surface for determining the design response spectrum, calculate the dynamic stress and strain for liquefaction assessment, and determine the earthquake load that will cause the instability of foundations or retaining structures.

Free field analysis is used to obtain the response of ground due to the vertical propagation of shear waves passing through the linear visco-elastic zone. The ground to be analyzed comprises a number of layers, which are infinite in the horizontal direction, and the semi-infinite base layer as shown in Figure 5.1.1. Each layer is assumed to consist of a homogeneous and isotropic material. The vibration is induced by the transmission and reflection of shear waves transferred through the ground in the vertical direction, and the displacements take place only in the horizontal direction. Therefore, the wave equation of Eq. (5.1.3) must be satisfied at every layer.

$$\rho \frac{\partial^2 u}{\partial t^2} = G \frac{\partial^2 u}{\partial x^2} + 2\xi G \frac{\partial^3 u}{\partial x^2 \partial t} \tag{5.1.1}$$

u is the horizontal displacement at the time t. ρ, G and ξ are mass density, shear elastic modulus and hysteretic damping ratio respectively.

Expressing the displacement function as a harmonic function of Eq. (5.1.2), and transforming Eq. (5.1.1) into a frequency domain result in the governing equation of Eq. (5.1.3). The stress-displacement relationship is given by Eq. (5.1.4).

$$u(x,t) = u(x,\omega)e^{i\omega t} \tag{5.1.2}$$

$$G^* \frac{\partial^2}{\partial x^2} u(x,\omega) - \rho\omega^2 u(x,\omega) = 0 \tag{5.1.3}$$

$$\tau(x,\omega) = G^* \frac{\partial}{\partial x} u(x,\omega) \tag{5.1.4}$$

where, $\tau(x,\omega)$ is shear stress, and $G^* = G(1+i2\xi)$ is complex shear elastic modulus.

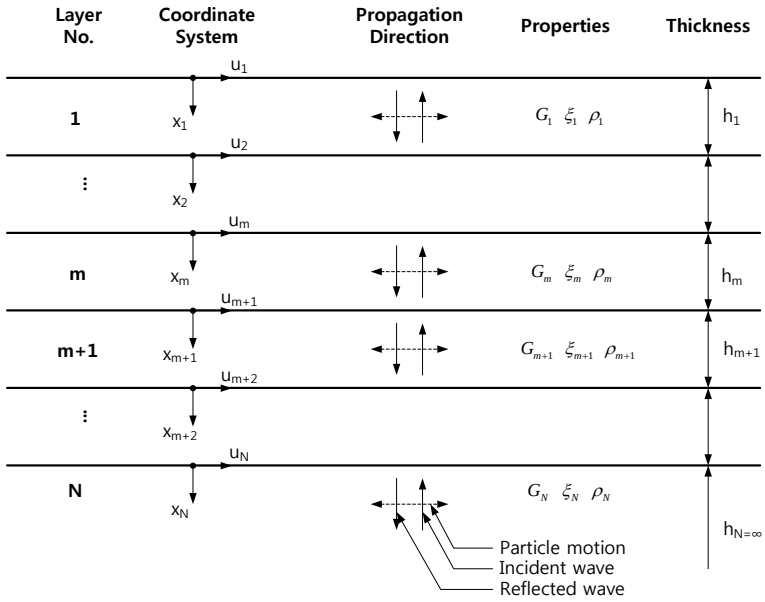


Figure 5.1.1 1-Dimensional free field analysis model

The free field ground is generally expressed as shown in Figure 5.1.1 in order to obtain the solution to one-dimensional wave transmission equation. The layer numbers are sequentially assigned from the ground surface. The response of the m-th layer is expressed as u_m. The response is then expressed as a function of x_m, which is the depth from the top of the m-th layer as shown in Eq. (5.1.5)-(5.1.6).

$$u_m(x_m,\omega) = A_m(\omega)e^{ik_m^* x_m} + B_m(\omega)e^{-ik_m^* x_m} \tag{5.1.5}$$

$$\tau_m(x_m,\omega) = ik_m^* G_m^* (A_m(\omega)e^{ik_m^* x_m} - B_m(\omega)e^{-ik_m^* x_m} \tag{5.1.6}$$

where, $k_m^* = \omega / V_{sm}^*$, $V_{sm}^* = \sqrt{G_m^* / \rho_m}$, and A_m and B_m are the layer response factors.

A_m is the elastic wave component transmitted upward, and B_m is the elastic wave component transmitted downward. The compatibility condition of Eq. (5.1.7) and force equilibrium must be satisfied at the boundary with the adjacent layers.

$$\begin{Bmatrix} u_m(x_m = h_m) = u_{m+1}(x_{m+1} = 0) \\ \tau_m(x_m = h_m) = \tau_{m+1}(x_{m+1} = 0) \end{Bmatrix} \; m = 1, 2, ..., (N-1) \tag{5.1.7}$$

Substituting Eq. (5.1.5) and Eq. (5.1.6) into Eq. (5.1.7) gives the relationship between the factors as shown in Eq. (5.1.8).

$$\begin{Bmatrix} A_{m+1} + B_{m+1} = A_m e^{ik_m^* h_m} + B_m e^{-ik_m^* h_m} \\ A_{m+1} - B_{m+1} = \dfrac{k_m^* G_m^*}{k_{m+1}^* G_{m+1}^*} A_m e^{ik_m^* h_m} - B_m e^{-ik_m^* h_m} \end{Bmatrix}, \; m = 1, 2, ..., (N-1) \tag{5.1.8}$$

Rearranging this to derive the relationship between the response factors of the adjacent ground layers, the recurrence equation of Eq. (5.1.9) is obtained.

$$\begin{aligned} A_{m+1} &= \frac{1}{2} A_m (1 + \alpha_m^*) e^{ik_m^* h_m} + \frac{1}{2} B_m (1 - \alpha_m^*) e^{-ik_m^* h_m} \\ B_{m+1} &= \frac{1}{2} A_m (1 - \alpha_m^*) e^{ik_m^* h_m} + \frac{1}{2} B_m (1 + \alpha_m^*) e^{-ik_m^* h_m} \end{aligned} \tag{5.1.9}$$

where,

h_m : Thickness of m th layer

α_m^* : Ratio of dynamic stiffness between the adjacent layers

$$\alpha_m^* = \frac{k_m^* G_m^*}{k_{m+1}^* G_{m+1}^*} \tag{5.1.10}$$

Since the shear stress at the ground surface is always zero, it is clear that $A_1 = B_1$ from Eq. (5.1.6). Therefore, the response factor of the m th layer can be calculated by sequentially applying the recurrence equation of Eq. (5.1.9) from the first layer to the m th layer.

$$A_m(\omega) = a_m(\omega)A_1(\omega)$$
$$B_m(\omega) = b_m(\omega)B_1(\omega) \tag{5.1.11}$$

where, $a_1 = b_1 = 1$

The transfer function, $H_{ij}(\omega)$, between the boundaries of the layer i and the layer j is expressed as,

$$H_{ij}(\omega) = \frac{u_i(\omega)}{u_j(\omega)} = \frac{a_i(\omega) + b_i(\omega)}{a_j(\omega) + b_j(\omega)} \tag{5.1.12}$$

When the transfer function, $H_{ij}(\omega)$ and the response, $u_j(\omega)$ at the boundary of the layer j are given, the response, $u_i(\omega)$ at the boundary of the layer i can be obtained as Eq. (5.1.13).

$$u_i(\omega) = H_{ij}(\omega)u_j(\omega) \tag{5.1.13}$$

When the frequency domain analysis is used, the effect of nonlinear behavior of the ground is generally considered by the equivalent linearization technique. The reason for using this equivalent linearization technique is that the frequency domain analysis basically assumes that the ground would behave linearly.

Since the constant stiffness (shear modulus) and damping are applied at all times in frequency domain analysis, the change in stiffness due to actual shear strain cannot be considered as shown in Figure 5.1.2. Therefore, the nonlinear behavior of the ground is considered by iteratively performing linear analysis with changing stiffness and damping ratio of the ground according to the shear strain obtained from the previous step, as shown in Figure 5.1.3. In order to determine the stiffness and damping, the effective shear strain is used, which is obtained by multiplying the maximum shear strain obtained from the previous step as shown in Eq. (5.1.5) by a constant value less than one (e.g., 0.65). The reason for the use of the effective shear strain is that the maximum shear strain induces larger deformation energy compared to the actual behavior, as shown in Figure 5.1.2. The method of considering the nonlinearity of ground by using such an equivalent linearization technique is summarized in Table 5.1.1.

$$\gamma_{\max} = \max_t \{\gamma_{oct}(t)\} \tag{5.1.14}$$

$$\gamma_{oct}(t) = \frac{1}{3}\sqrt{\varepsilon_x^2(t) + \varepsilon_z^2(t) + \{\varepsilon_x(t) - \varepsilon_z(t)\}^2 + 6\gamma_{yz}^2(t) + 6\gamma_{xz}^2(t)} \tag{5.1.15}$$

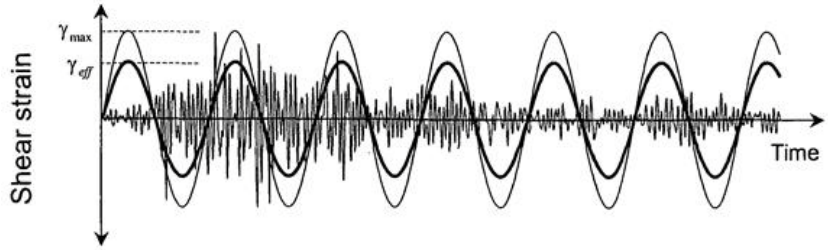


Figure 5.1.2 Difference between maximum shear strain and effective strain

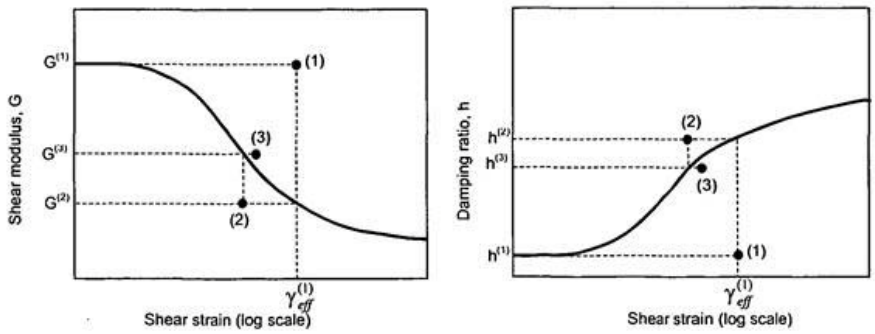


Figure 5.1.3 Process of convergence for nonlinear shear modulus and damping factor by equivalent linearization technique

Table 5.1.1 Procedures for free field analysis using equivalent linearization technique

1.	Define the initial values of the shear modulus (G) and damping ratio (h) of each layer. In general, values at very small strain are used.
2.	Calculate the maximum shear strain $\gamma_{\max}$ at each layer by performing free field analysis using the initial values.
3.	Calculate the effective shear strain γ_{eff} at each layer.

$$\gamma_{eff} = R_{\gamma} \times \gamma_{\max}$$

where, R_{γ} is the ratio of the effective shear strain to the maximum shear strain for which 0.65 is used or the following value based on the magnitude of earthquake, M is used (Seed & Sun, 1992).

$$R_{\gamma} = \frac{M-1}{10}$$

4. Determine the shear modulus, G and h according to the $G/G_{\max} - \gamma_{eff}$ curve and $h - \gamma_{eff}$ curve of each layer using the effective shear strain, γ_{eff} of each layer.
5. Repeat the steps from 2 to 4 until G and h converge. In general, a relative error less than 5% is considered to have converged, which takes place within 5 iterations.

5.2 2-Dimensional Dynamic Analysis (Finite Element Method)

5.2.1 Introduction

The biggest difference between a soil-structure interaction problem and a general structural dynamics problem is the radiation damping, which stems from the infiniteness of the ground. While the general damping property dampens the structural motion by material friction, the radiation damping reduces the structural kinetic energy by releasing the wave energy into the infinite domain of the ground.

Radiation damping is included in the equation of motion as a damping term, and its magnitude is dependent on the shape of the wave propagating outward. Since the wave shape can be easily modeled in a frequency domain, frequency domain analysis can be efficiently used. In general, ground materials are basically inhomogeneous, and the mechanical behavior of the ground is highly nonlinear.

In order to accurately analyze soil-structure interaction problems, the important characteristics of radiation damping and nonlinearity must be simultaneously considered. Therefore, frequency domain analysis is used to readily model the radiation damping, and the material nonlinearity is analyzed using the equivalent linearization method.

The analysis process entails the use of the standard frequency domain analysis using the FFT (Fast Fourier Transform), and the equation of motion composes an integrated ground-structure system with the free field response as the input motion. As such, the ground response and the structure response can be calculated in one operation. The interpolation method of transfer function for a single degree of freedom system is used to reduce the number of frequencies for finding solutions to the equation of motion. This method is used to obtain the interpolated value of the transfer function based on the solutions of the transfer functions for two consecutive frequencies. Therefore, selecting the basic frequencies is very important, and a sufficiently high cut-off frequency needs to be defined to secure the validity of the ground motion analysis.

5.2.2 Equation of Motion

From the results of free field analysis as mentioned above, the analysis is performed by inputting the seismic response at the bedrock as the input motion in the equation of motion of Eq. (5.2.1).

$$[M]\{\ddot{u}\}+[K]\{u\}=-\{m\}\ddot{y}+\{F\}-\{T\} \tag{5.2.1}$$

where,

$\{u\}$	: Relative displacement vector of each node at the base of the model
$[M]$	: Mass matrix of the two-dimensional soil-structure system with a unit thickness
$[K]$	: Complex stiffness matrix of the two-dimensional soil-structure system with a unit thickness, which includes the damping effect
$\{m\}$	: Mass vector in the direction of input motion
$\ddot{y}$	: Input motion acting on the base of the model
$\{V\}$	: Force generated at the viscous boundary of three-dimensional direction, as given by Eq. (5.2.2)

$$\{V\}=\frac{1}{L}[C](\{\dot{u}\}-\{\dot{u}_f\}) \tag{5.2.2}$$

L above represents the thickness of the plane system, $[C]$ is the diagonal matrix exhibiting the characteristic of free field, and $\{\dot{u}_f\}$ is the velocity vector of the free field.

$\{F\}$ in Eq. (5.2.1) is characterized as the load acting on the vertical faces of the soil-structure system, as given by Eq. (5.2.3).

$$\{F\}=[G]\{u_f\} \tag{5.2.3}$$

$[G]$ is the complex stiffness matrix of free field, and $\{u_f\}$ is the displacement vector of free field. $\{T\}$ in Eq. (5.2.1) is the force generated by the horizontal transfer of wave energy, which is expressed as the transmitting boundary condition.

$$\{T\} = ([R]+[L])(\{u\}-\{u_f\}) \tag{5.2.4}$$

where, $[R]$ and $[E]$ are the stiffness matrices based on the frequencies of boundary conditions developed by Lysmer Drake and Wass.

Assuming the input motion $\ddot{y}(t)$ in Eq. (5.2.1) as the sum of a finite number of harmonic motions as shown in Eq. (5.2.5), frequency domain analysis is performed.

$$\ddot{y}(t) = \mathrm{Re}\sum_{s=0}^{N/2}\ddot{Y}_s e^{i\omega_s t} \tag{5.2.5}$$

where, N is the number of points representing the input motions. Therefore, the response of a soil-structure system is given as Eq. (5.2.6)

$$\{u\} = \mathrm{Re}\sum_{s=0}^{N/2}\{U\}_s e^{i\omega_s t} \tag{5.2.6a}$$

$$\{u_f\} = \mathrm{Re}\sum_{s=0}^{N/2}\{U_f\}_s e^{i\omega_s t} \tag{5.2.6b}$$

Substituting Eq. (5.2.6) and Eq. (5.2.2) into Eq. (5.2.1), and rearranging it provide Eq. (5.2.7).

$$\begin{aligned}&([K]+[R]_s+[L]_s-\omega_s^2[M])\{U\}_s\\&=-\{m\}\ddot{Y}_s+([G]+[R]_s+[L]_s)\{U_f\}_s\end{aligned} \tag{5.2.7}$$

In Eq. (5.2.7), $\{U_f\}_s$ can be expressed as Eq. (5.2.8), and $\{A_f\}_s$ is the transfer function with respect to the input motion at the base.

$$\{U_f\}_s = \{A_f\}_s\ddot{Y}_s \tag{5.2.8}$$

Substituting Eq. (5.2.8) into Eq. (5.2.7), and rearranging it result in Eq. (5.2.9). $[K]_s$ is the stiffness matrix with respect to the frequency of the system, and $\{P\}_s$ is the load vector for the input motion of $\ddot{Y}_s = 1$.

$$[K]_s\{U\}_s = \{P\}_s \ddot{Y}_s \tag{5.2.9a}$$

$$[K]_s = [K] + [R]_s + [L]_s - \omega_s^2[M] \tag{5.2.9b}$$

$$\{P\}_s = ([G] + [R]_s + [L]_s)\{A_f\}_s - \{m\} \tag{5.2.9c}$$

Solving Eq. (5.2.10), which is obtained by dividing both sides of Eq. (5.2.9) by $\ddot{Y}_s$, $[K]_s$ and $\{P\}_s$ are a function of frequency, ω, and $\{A\}_s$ is also a function of frequency.

$$[K]_s\{A\}_s = \{P\}_s \tag{5.2.10}$$

$\{A\}_s$ is thus obtained from Eq. (5.2.10), and $\{u\}$ of Eq. (5.2.6a) can be now written as Eq. (5.2.11).

$$\{u\} = \mathrm{Re}\sum\{A\}_s \ddot{Y}_s e^{i\omega_s t} \tag{5.2.11}$$

Also, the response in a time domain can be obtained by inverse transformation of Eq. (5.2.7) into a time domain using the FFT method.

5.3 Transmitting Boundary

The fact that the ground exists in infinity, it is difficult to accurately simulate the actual soil-structure interaction by a 2-dimensional model. Therefore, the boundaries must be modeled at appropriate positions to reflect as closely to the actual site conditions as possible.

The boundary conditions in ground modeling are divided into element boundary, viscous boundary and transmission boundary conditions. The element boundary condition is further subdivided into free ends to which the force from earthquake response loading at the boundary for the free field is assigned, fixed ends to which the displacement is assigned, and rotating ends representing the horizontal and vertical rotating ends. The element boundary condition can sufficiently consider the effects of earthquake waves for the free field, but cannot consider the effects of waves reflected from the foundation slab. Also, such effects increase as the boundary is located closer to the foundation slab.

In order to overcome the shortcomings of the element boundary condition, Kuhlemeyer, Ang, Newmark, et al. developed the viscous boundary condition that can absorb the material waves at a constant angle to the boundary. Even then, the viscous boundary condition cannot completely consider the effects of complex surface waves, which necessitates adequate separation between the boundary and the foundation slab like the element boundary.

The transmitting boundary condition has complemented certain limitations of the viscous boundary condition. It can consider the effects of practically all material waves and surface waves. The soil layers in the horizontal direction are represented by springs and dampers as a function of frequency. Since it is generally assumed that the property of each soil layer in the horizontal direction is homogeneous, satisfactory results can be obtained for cases even where boundary conditions are assigned to the structure itself. However, it is still effective to maintain certain distance between the boundary and the foundation slab in order to accurately consider the change in properties due to the strains in the horizontal direction.